The Already-Unified $\mathfrak{sl}(4,\mathbb{R})$ Invariant Gauge Theory of General Relativity

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Abstract

The $\mathfrak{sl}(4,\mathbb{R})$ algebra as a background-dependent representation of quantum fields is a basis for unification as a gauge invariance of the metric in General Relativity, a background-independent theory. No changes are made to the Einstein field equations. This algebra contains both fermionic and bosonic fields as well as gravitational fields. Mass parameters are constrained by the volume integral of the trace of the stress-energy tensor, providing a non-linear eigenvalue equation for mass values. An example is provided for static, spherically symmetric gravity. In different regimes this gives normal Schwarzschild gravity, black holes without singularity, galaxy rotation curves. Dark Matter and Dark Energy entail.

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1. *Introduction*

To completely understand and control gravity, its relationship with the other forces and the quantum realm must be understood — beyond the assertion that all forms of energy generate gravity, one that has not been completely tested. This is challenging for several reasons. The action-at-a-distance problem has been solved for gravity through a dynamic spacetime. This is not so for the other forces, where fields still perform that function. It would seem that the elimination of those fields (replacing them with a spacetime construct), is necessary to complete the task of geometrizing all the forces. Even so, there are other issues. General Relativity (GR) is a theory of spacetime not of matter; both the mechanism by which matter affects the curvature, and how curvature determines the motion of matter, are left unspecified. A closely related issue is that of Mach's principle: the details of the interaction of matter with spacetime can hardly be understood without specifying the origin of inertia. Perhaps most intractable are the differences between GR and Quantum Theory (QT). GR as a coordinate invariant theory is called background independent, while QT "depends" upon a specific choice of metric. GR is a non-linear theory, while QT is based upon the superposition principle, which requires linearity.

So it seems a geometric view of matter is required as well. Such a construction would hopefully eliminate the singularities in black holes. This is in fact what happens.

The problem of the lack of a fundamental description of matter also plagues QT. Beyond quantum numbers and symmetry groups, matter is treated simply as point particles. This is the source of the infinities that plague QT [1], notwithstanding renormalization.

These problems are all clearly interdependent. That is why a complete understanding of gravity has been a refractory endeavor. Fortunately, there is a simple solution.

2. Methodology

A subtle yet profound change to the ontological basis of physics can both lead the way to a unified field theory and shed light on the epistemological differences between GR and QT. There is a simple framework for unification that is testable, refutable, and leaves the Einstein Field Equation (EFE), Maxwell Equations (ME) as well as QT intact. This is a geometrization of quantum fields rather than a quantization of gravitational fields; gravity naturally takes its place as a quantum field among the others.

The key to understanding gravity is, strangely enough, understanding Electromagnetism (EM). The clues provided by the ME, using simple reasoning, define gravity's relationship to the other forces, and to GR, as well as GR's relationship to QT. The result is a simple gauge invariance of GR with a remarkable list of consequences. That is the invariance of the 4-volume under the Lie algebra $\mathfrak{sl}(4, \mathbb{R})$.

The following set of deductions and inferences provides a compelling argument for this already-unified gauge field theory. The reasoning is based upon two principles that are not entirely independent. They are actually inferred from the theory, not the other way around.

1. It is necessary to mitigate action-at-a-distance to fit the other forces into GR. The use of the word mitigate above is literal, to lessen. Assuming matter at a distance cannot interact without fields yields a contrapositive; if material particles do interact without fields, then they must not be "at a distance". There is only one way this can happen: matter itself must extend through spacetime, as part of the same continuum, so that one particle can smoothly meld into another. So in order to eliminate the field concept, a particle must be made up of the same "stuff" as spacetime with most of it fairly well localized to appear as a particle. This is a significant change to the ontology of physics.

Two particles can therefore interact using a field as an intermediary, or a dynamical spacetime. The field concept becomes superfluous if it assumed that a spacetime displacement field solves the same equations locally as the locally derived field theories. This is shown below and is the implementation of what has been called the *Maxwellian dream* [1]. A spacetime composed of the tails of all the matter distributions in the universe has the same properties as the Dark Matter (DM) — empty space is not empty. This also explains why ΛCDM, despite it shortcomings, works as well as it does given the clumpiness of space.

2. The field should be free from singularities.

   The presence of singularities in GR is unacceptable [2]; an acceptable theory has to work everywhere. Infinities are unmeasurable. Particles must be represented by a finite matter field in some finite region. The observables of spacetime are distance and duration. These are specified by the metric tensor. Therefore, for any real configuration of matter there must exist a coordinate system that results in measurable intervals everywhere. As will be shown, the removal of singularities is an automatic byproduct of the gauge theory.

3. The Gauge Field

GR is a gauge theory. The gauge invariance of GR is the invariance of the spacetime interval under coordinate transformations. In all the examples analyzed the gauge field has zero trace. Therefore the gauge fields will all be assumed to be based upon the gauge invariance of the 4-volume, that is, the Lie group $SL(4,\mathbb{R})$. It has been shown that all the Lie groups of the SM can be found from the corresponding algebra of $\mathfrak{sl}(4,\mathbb{R})$ [3].

The Maxwell equations are analogous to the equations of fluid flow, complete with sources, sinks and vortices. This was noted by Maxwell early on and there were attempts to mechanize the field with a quasi-elastic ether model [4]. The approach was to assume space contained some kind of "aether" that could flow or spin. These ideas were unworkable. Therefore the analogy is either an accidental coincidence, or it represents some other kind of motion. There is only one other possibility for such a displacement field. It is that *the Maxwell equations represent a transformation of spacetime points themselves, rather than some substance occupying spacetime points.* Simply accepting it as a "field" admits that the structure of the equations is a coincidence. Such an acceptance also introduces a new elementary object that requires its relationship to gravity be separately defined, unnecessarily complicating the ontology — Occam's razor. After all, EM fields can only be "detected" by the charges that are supposedly their sources. The fact that they can carry energy and momentum simply translates to spacetime displacements carrying them instead. The transformation of events can be described mathematically in the same way as that of a deformable physical medium. Consider an infinitesimal displacement, $\boldsymbol{\xi}$, in the neighborhood of a small volume element in a 3 dimensional Euclidean space. It is composed of a rotation, a compression (extension or shear), and a translation [5]. Expressing this in terms of the symmetry properties of an infinitesimal displacement field in 4 dimensions now,

$$\xi^{\mu}_{;\nu} dx^{\nu} = \frac{1}{2} g^{\mu\lambda} \left(\xi_{\lambda;\nu} + \xi_{\nu;\lambda}\right) dx^{\nu} + \frac{1}{2} g^{\mu\lambda} \left(\xi_{\lambda;\nu} - \xi_{\lambda;\mu}\right) dx^{\nu} = g^{\mu\lambda} \sigma_{\lambda\nu} dx^{\nu} + g^{\mu\lambda} \alpha_{\lambda\nu} dx^{\nu} \tag{1}$$

with σ the symmetric tensor and α antisymmetric. Finite displacements can then be represented as exponentiated displacements as usual,

$$ds^2 = \bar{g}_{\mu\nu} \left(\left(e^{\boldsymbol{\zeta}}\right)^{\mu}_{\lambda} dx^{\lambda} \right) \left(\left(e^{\boldsymbol{\zeta}}\right)^{\nu}_{\tau} dx^{\tau} \right) = \left(\left(e^{\boldsymbol{\zeta}}\right)^{\mu}_{\lambda} \bar{g}_{\mu\nu} \left(e^{\boldsymbol{\zeta}}\right)^{\nu}_{\tau} \right) dx^{\lambda} dx^{\tau} = g_{\lambda\tau} dx^{\lambda} dx^{\tau} \tag{2}$$

Thus the displacements can be considered either as new coordinates for the points using the old metric, $\bar{g}$, or as a new, transformed metric, g, using the old coordinates. The gauge field, $\boldsymbol{\zeta}$, transforms the metric, and as argued below, contains all the forces.

The transformed metric using the old coordinates can be thought of as the consequence of using the "wrong" coordinates, that consequence giving rise to "fictitious" or inertial force fields, the usual view of gravity. This gauge transformation of the metric tensor is a type of factorization of the metric, but based on general tensors rather than tetrads. The transformations among the -variant forms of $\boldsymbol{\zeta}$ and its covariant derivatives still use the base space metric since they are measured in the local tangent space to the manifold, and with respect to the old coordinates.

All the formulations of physical laws using the field $\boldsymbol{\zeta}$ and its tangent space metric are therefore "background dependent" [6]; they rely on a background metric that may vary from event to event. Their phenomenologies are derived in that local space, depend on its metric, and as such cannot be expected to have a generally covariant form. *They are flat-space laws.* However, Eq. (2) defines their relationship to the exact metric tensor and therefore defines their participation in a generally covariant "background-independent" theory, GR. That theory now involves all the forces, at all scales, classical as well as quantum fields.

The flat space laws are subject to quantization which relies on their background dependency. These quantum fields can be mapped into $\boldsymbol{\zeta}$ based upon their analytic and geometric properties, when expressed as functions of the coordinates [3]. This gives them a classical picture. Although the terms "field" and "displacement field" are being used, no new fields are being introduced; these mathematical fields just describe the displacement of events from their base-space coordinate locations in spacetime. Now the connections among the Maxwell equations, spacetime flows and rotations, and the metric tensor can be specified.

The antisymmetric part of the displacement field in Eq. (1) represents rotations and flows of events with respect to the base space coordinate system. It is also an exact tensor so automatically satisfies 2 of the Maxwell equations. It is therefore taken to be (proportional to) the EM field, F, which is defined in terms of the vector potential,

$$\alpha_{\mu\nu} = \frac{1}{2}\left(\xi_{\mu,\nu} - \xi_{\nu,\mu}\right) = \frac{1}{2}\xi_{\,[\mu,\nu\,]} \equiv f_{\mu\nu} \propto F_{\mu\nu} = \eta f_{\mu\nu},\ F_{\mu\nu} = \phi_{\mu,\nu} - \phi_{\nu,\mu},\ \phi^{\mu} = \left(A^0, A^{\iota}\right) \quad (3)$$

4. The Already-Unified Gauge Field Hypothesis

Consider the generalization of Eq. (2) where the gauge field ζ is any second rank tensor. ζ can be decomposed into a tensor with zero divergence and one with zero curl (antisymmetrized derivative), and this process continuing with the resulting vector and scalar fields. It is the main hypothesis here that these displacement fields, ζ, are

$$\boxed{\zeta_{\mu\nu} = \chi_{\mu\nu} + \left(\phi_{\mu;\nu} + \phi_{\nu;\mu}\right) + \lambda_{,\mu;\nu} + f_{\mu\nu}}, \quad (4)$$

where both the tensor field χ and the vector field ϕ have zero divergence, λ is a scalar field and f is the EM field. The symmetric fields generate gravity; scalar, vector and tensor aspects. The antisymmetric vector field is EM. For a given configuration of matter the equations of these fields are known: Klein-Gordon, Maxwell, Dirac, Proca, etc. Since these fields are also linear operators, and they satisfy the necessary $\mathfrak{sl}(4,\mathbb{R})$ commutation relationships, they are also quantum fields. They are what appear in the local space as force fields. More generally all linear combinations of the generators of the $\mathfrak{sl}(4,\mathbb{R})$ algebra are possible, including real spinor representations of the group [3]. *This is the implementation of the first principle.* The physical significance of this invariance under the group $SL(4,\mathbb{R})$ is that the aether seems like an elastic material. Spacetime can bend and stretch but not tear. The accumulation at one event must be compensated at another event. This crucial fact is at the heart of both DM and Dark Energy (DE).

5. Example - Spherically Symmetric Gravitational Field

The EM field can be expressed as an antisymmetric tensor and gravity is the symmetric part. This then gives the two types of fields their known properties: EM, the antisymmetric part cannot, by itself cause gravity (it is gauge dependent), nor be transformed away by any real coordinate change. On the other hand gravity, the symmetric part, can be transformed away, locally, leading to the known universality of free-fall. Also gravity is often depicted as stretches or compressions of the "fabric" of space and time. Identifying the symmetric field with gravity is therefore consistent with the prevalent picture.

In all cases the methodology is the same. The gauge field is the local solution to the appropriate equation of state. This gives the metric from Eq. (2). That metric is put in the EFE giving the matter tensor. The trace of the matter tensor is the scalar invariant mass density. The density is integrated over the 4-volume to give the mass. This is very different from the way solutions to the EFE are normally obtained. The mass appears both in the local gauge field solution and integrated density tensor. They must be equal. This is a constraint on allowable masses in general. It is hypothesized that this will generate the spectrum of masses of elementary particles for the appropriate gauge fields. This is true for the spherically symmetric (spin zero), scalar masses shown next. The masses are equal but unconstrained in this case. In any case this shows that masses are completely composed of gravitational fields and the EFE is interpreted an identity.

The gauge field for the Schwarzschild-like solution to the EFE is given in spherical coordinates by the diagonal tensor $\{\Phi(r), -\Phi(r), 0, 0\}$. Locally, this is a single component, i.e., scalar field, Φ (a tensor field with one independent component). Its equation of state is the Klein-Gordon (K-G) equation. The spherically symmetric, static solution is a Yukawa potential. The expected mass enclosed is the integral of its square up to some point r. This gives a gauge field of 2 terms: a Yukawa term and a classical gravitational potential for the mass m. It is also obtained by solving the K-G equation in the classical gravitational potential of the mass by adding an interaction term to the Lagrangian density for the K-G equation [7]. This is a material particle (region of spacetime) interacting with its own gravitational field with coupling constant k. *It is this interaction that couples the inertial mass to the gravitational mass of matter.*

$$\mathcal{L} = \frac{1}{2}\Phi'(r)^2 - \frac{1}{2}k^2\Phi(r)^2 + \mathcal{L}_{\text{int}}, \quad \mathcal{L}_{\text{int}} = V(r)\Phi(r), \quad V(r) = -k^2\frac{m}{r}, \tag{5}$$

yielding

$$\Box\Phi(r) + k^2\Phi(r) - V(r) = 0 \Rightarrow \Phi(r) = -m\frac{\left(1 - e^{-kr}\right)}{r}. \tag{6}$$

If the Yukawa term were used by itself, the result is equal and opposite amounts of positive and negative gravitational energy, giving a net of zero — an interesting result in its own right. This is analogous to the way the Higgs field couples to the electroweak field.

For most values of mass and radius this gravitational field is indistinguishable from that of the Schwarzschild solution. The differences are as follows. This field is never zero, it is only approached asymptotically. Spacetime, gravity and the mass are all the same thing and nonzero everywhere. The Schwarzschild solution is for $G = 0$ in the EFE. The space is empty outside the mass. The solution is only valid outside the mass. This metric is finite at the origin. There is no singularity and no event horizon for such "red" holes.

It is very difficult to discern this solution (red hole) from a black hole. As shown in Fig. 1 for the Innermost Stable Circular Orbit (ISCO), any phenomenon occurring at r_b around a black hole occurs at r_r around a red hole, a somewhat smaller radius.

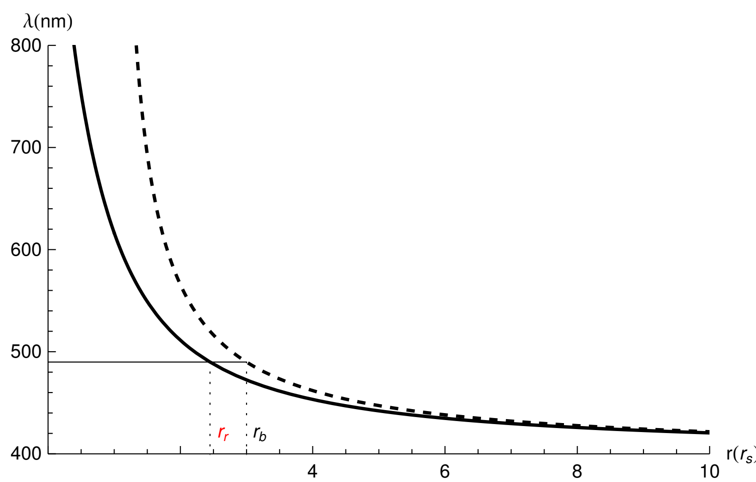

FIG. 1. ISCO red shift of 400 nm light, black hole (dotted) vs. red hole models.

For galaxy-sized masses, there is a characteristic hump in the enclosed-mass vs radius curve, similar to that found in galaxy rotation curves as in Fig. 2. It is known that rotation curves lose their dependence on radius and depend only on the enclosed mass [8]. It is caused by a region of negative energy density due to the Yukawa term as seen in Fig. 3. This flattens out the rotation curve like DM, and at far distances the negative ratio of energy to pressure looks like DE.

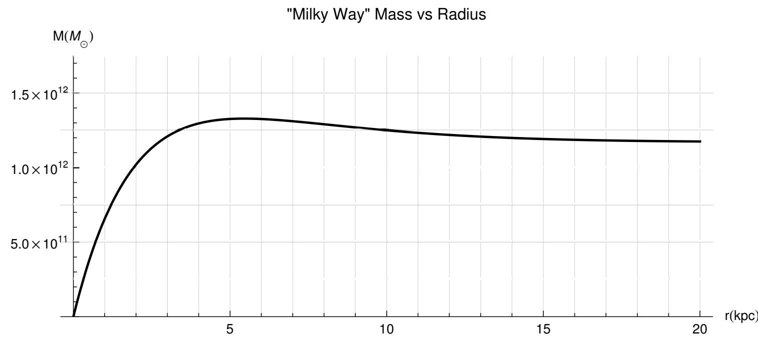

FIG. 2. Mass profile for the complete gauge field for a Milky Way model ($m = 5.6 \times 10^{-5}$ $kpc, k = 2.2/6$ kpc^{-1}).

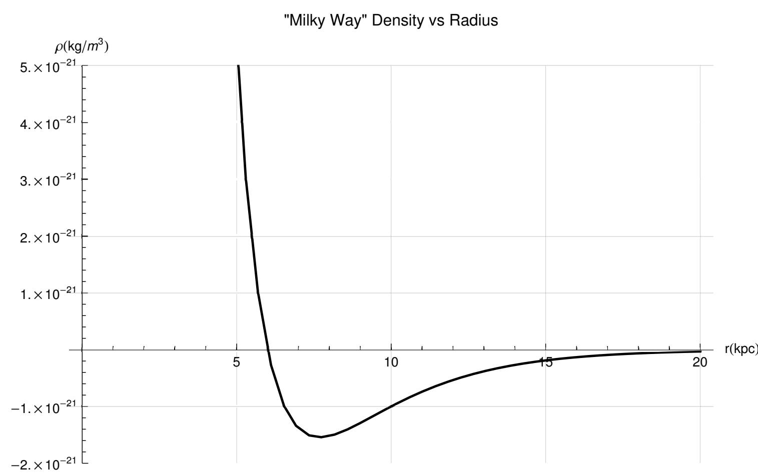

FIG. 3. Scalar density for the complete gauge field for a Milky Way model ($m = 5.6 \times 10^{-5}$ $kpc, k = 2.2/6)$ kpc^{-1}).

6. Conclusion

This theory is a complete unification of gravity and the other forces. There are many more consequences of this theory. Electric fields are a source of gravity, magnetic fields, negative gravity. EM planes waves follow geodesics, but are not a source of gravity. Gravitational radiation is a source of gravity in covariant form, with a matter tensor identical to the TT-mode Isaacson tensor of the linearized theory. Cosmological consequences are many.

[1] E. Pessa, arXiv: General Physics (2009).

[2] J. Earman, in *The Expanding Worlds of General Relativity*, edited by H. Goenner, J. Renn, J. Ritter, and T. Sauer (Birkhauser Boston, 1999) pp. 235+.

[3] R. A. Wilson, International Journal of Geometric Methods in Modern Physics https://doi.org/10.1142/S021988782250164X (2022).

[4] A. Sommerfeld, Mechanics of deformable bodies, 2nd ed. (Academic Press, 1950) Chap. 15.

[5] H. von Helmholtz, Crelles J. **55** (1858).

[6] D. Rickles, Philosophy and Foundations of Physics **4**, 133 (2008).

[7] D. Tong, Lectures on quantum field theory (David Tong, 2006) Chap. 1, p. 8.

[8] S. McGaugh, Galaxies **2**, 601 (2014).